Practice Problems for the Final Exam

Q1. Determine whether each of the following series are convergent or divergent. You may use any applicable test covered in this course.

(1) $\sum_{n=0}^{\infty} \arctan(n^2)$ div. by DT

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ conv. by LCT with $\sum \frac{1}{n^2}$

(3) $\sum_{n=0}^{\infty} \frac{(n+1)(3^2-1)^n}{3^{2n}}$ conv. by Ratio Test

(4) $\sum_{n=0}^{\infty} \frac{2n^2+3n+4}{\sqrt{n^5+6n^2+7}}$ div. by LCT with $\sum \frac{1}{\sqrt{n}}$;

(5) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ div. by CT with $\sum \frac{1}{n}$; use $\ln(n) > 1$

(6) $\sum_{n=1}^{\infty} \ln\left(\frac{2n^2+3}{n^2+n}\right)$ div. by DT;

(7) $\sum_{n=0}^{\infty} \frac{(-3)^n}{2^{2n+1}}$ conv. by $|r| = \frac{3}{4} < 1$;

(8) $\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ conv. by LCT with $\sum \frac{1}{n^2}$;

(9) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^n}$ conv. by Root Test;

(10) $\sum_{n=0}^{\infty} \frac{(-9)^n}{6^{n+1}}$ div by $|r| = \frac{9}{6} \geq 1$;

(11) $\sum_{n=0}^{\infty} \frac{2^n}{4^n+1}$ conv. by LCT with $\sum \frac{2^n}{4^n}$;

(12) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n(2^n+1)}$ div. by Ratio Test;

(13) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ conv. by Ratio Test;

(14) $\sum_{n=5}^{\infty} \frac{2n^2-1}{n^4-n^3+n^2+1}$ conv. by LCT with $\sum \frac{1}{n^2}$

(15) $\sum_{n=0}^{\infty} \frac{1}{n!}$ conv. by Ratio Test;

(16) $\sum_{n=1}^{\infty} \frac{1}{n^{\sin(1)}}$ div by $p = \sin(1) \leq 1$

(17) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln(n))^2}$ conv. by AST;

(18) $\sum_{n=0}^{\infty} \frac{4^{n-2}(3^n)}{8^{2n+1}+2}$ conv. by LCT with $\sum \left(\frac{12}{16}\right)^n$

(19) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$ conv. by AST or by p-series

(20) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ conv. by Integral Test;

(21) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ conv. by AST; - conditionally conv;

(22) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n+2}$ div. by DT or by LCT;

(23) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ conv. by AST;

(24) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}$ conv. by Ratio Test;

(25) $\sum_{n=3}^{\infty} \left(\ln(n) - \ln(n+1)\right)$ div as a telescoping series;

(26) $\sum_{n=0}^{\infty} \frac{n!}{3^{n+1}}$ div. by the Ratio Test/DT;

(27) $\sum_{n=1}^{\infty} \frac{\cos^2(n)\sqrt{n}}{n^2}$ conv. by CT with $\sum \frac{1}{n\sqrt{n}}$;

(28) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$ conv. by AST;

(29) $\sum_{n=1}^{\infty} \frac{(5n^2+5)(-9)^n}{10^{n+3}}$ conv. by Ratio Test;

(30) $\sum_{n=1}^{\infty} \left(\frac{n+9}{5n+6}\right)^n$ conv. by Root Test;

(31) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ conv. by Ratio Test;

(32) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ div. by Integral Test;

(33) $\sum_{n=2}^{\infty} \frac{n^3+n^2+n+1}{2n^2-3n+4}$ div. by DT or by LCT;

(34) $\sum_{n=2}^{\infty} \cos\left(\frac{\pi}{n}\right)$ div. by DT;

(35) $\sum_{n=2}^{\infty} \frac{(\ln(n))^4}{n+3}$ div. by CT with $\sum \frac{1}{n}$; use $\ln(n) > 1$;

(36) $\sum_{n=1}^{\infty} \left(\frac{2n^2+6n}{8n^4+4}\right)^3$ conv. by LCT with $\sum \left(\frac{n^2}{n^4}\right)^3 = \sum \frac{1}{n^6}$;

Q2. Give approximations of the following series up to the specified accuracy. Your answer must be justified by some estimation theorem covered in class.

(1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ with error ≤ 0.005.

Not in exam. ~~**(2)** $\sum_{n=1}^{\infty} \frac{1}{n^3}$ up to 3 decimal places.~~

(3) $\sum_{n=1}^{\infty} \frac{(-3)^{2n+1}}{(2n+1)!}$ up to 3 decimal places.

Not in exam ~~**(4)** $\sum_{n=1}^{\infty} ne^{-n}$ with error $\leq 10^{-3} = 0.001$.~~

Q3. Find a power series representation for the following functions and identify an open interval of convergence (i.e. don't bother with the endpoints, i.e. giving the radius of convergence R suffices).

(1) $f(x) = \arctan(x)$

(2) $g(x) = \frac{1}{(1-x)^2}$

(3) $h(x) = \frac{x^2}{1+x^2}$

(4) $a(x) = x^2\sin(x)$

(5) $b(x) = \int_0^2 \cos(x^2)\,dx$

(6) $c(x) = \frac{d}{dx}\left(\frac{x^2}{1-x^3}\right)$

Q4. Determine the radius of convergence R and the interval of convergence I for the following power series.

(1) $\displaystyle\sum_{n=0}^{\infty}(3x-2)^n$ **(2)** $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n+1}$ **(3)** $\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n+1}}{(n+1)!(2n+1)}$

Q5. Approximate the following integrals up to the specified accuracy. Your answer must be justified by some estimation theorem covered in class.

(1) $A_1 = \displaystyle\int_0^1 x^2 e^{-x^2}\,dx$ up to 3 decimal places.

(2) $A_2 = \displaystyle\int_0^1 \sin(x^2)\,dx$ up to 3 decimal places.

(3) $A_3 = \displaystyle\int_0^2 \cos(x^2)\,dx$ with error ≤ 0.001.

(4) $A_4 = \displaystyle\int_0^{\frac{3}{4}} \arctan(x^2)\,dx$ with error ≤ 0.0001.

(5) $A_5 = \displaystyle\int_0^{\frac{2}{3}} x\ln(1+x^3)\,dx$ up to 5 decimal places.

(6) $A_6 = \displaystyle\int_0^1 e^{-x^2}\,dx$ with error ≤ 0.0005.

Q6. Find the 5th order Taylor polynomials of the following functions about $x=0$:

(1) $f(x) = \sqrt{1+x}$; (2) $g(x) = \sqrt{1-x}$; (3) $h(x) = \arctan(x)$;

Q7. Using Taylor's Inequality / Lagrange's Remainder Theorem, find minimal N such that $T_N(x_0)$ approximates $f(x_0)$ within the specified error range where $T_N(x)$ is the N^{th} order Taylor polynomial of $f(x)$ about $x=0$;

(a) $\cos(1)$ with $f(x)=\cos(x)$ and error ≤ 0.0005;

(b) $\sin(\frac{1}{2})$ with $f(x)=\sin(x)$ and error ≤ 0.00005;

(c) $\sqrt{e}$ with $f(x)=e^x$ and error ≤ 0.000005;

(d) e with $f(x)=e^x$ and error ≤ 0.000005;

(e) e^2 with $f(x)=e^x$ and error ≤ 0.0001;

Answers.

$N=6$;
$N=5$;
$N=6$;
$N=9$;
$N=11$;

Q2: Approximate sums.

(1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ with error ≤ 0.005;

Let $b_n = \frac{1}{n^2+1}$; Show that series converges by AST. When $N=14$: $b_{15} = 0.0044 < 0.005$; ← This is the minimum guaranteed by ASET.

By ASET: $S_{14} = \sum_{n=0}^{14} \frac{(-1)^n}{n^2+1} = \underline{0.638}$ is accurate to within 0.005;

(2) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ up to 3 decimal places. Use area-corrected approximations.

Let $f(x) = \frac{1}{x^3}$;
(1) For $x \geq 1$, $f(x)$ is positive;
(2) $f(x)$ is continuous for all $x \in \mathbb{R}$ with $x \neq 0$;
(3) $f'(x) = (-3)x^{-4}$ is negative for $x \geq 1$; $\therefore f(x)$ is decreasing for $x \geq 1$;

Then, $\lim_{x\to\infty} \int \frac{1}{x^3}dx = \lim_{x\to\infty} \int x^{-3}dx = \lim_{x\to\infty} \left(\frac{1}{-2}\right)x^{-2}dx = 0$; $\therefore 0 < S - U_N < f(N+1)$;

Find $x \geq 1$ such that $f(x+1) < \frac{1}{2}(10^{-3}) = 0.0005$;

$f(x+1) = \frac{1}{(x+1)^3} = 0.0005$; $(x+1)^3 = \frac{1}{0.0005}$; $x+1 = \left(\frac{1}{0.0005}\right)^{\frac{1}{3}} \approx 13$; $x = 13-1 = 12$;

Choose $N = 12$;

Then, $U_{12} = \sum_{n=1}^{12} \frac{1}{n^3} + \int_{13}^{\infty} \frac{1}{x^3}dx = \sum_{n=1}^{12} \frac{1}{n^3} + \left[(0) - \left(-\frac{1}{2}\right)(13)^{-2}\right]$

$$\boxed{= \sum_{n=1}^{12} \frac{1}{n^3} + \frac{1}{2(13)^2} = 1.202}$$ rounded to 3 decimal places

(3) $\sum_{n=1}^{\infty} \frac{(-1)^n(3)^{2n+1}}{(2n+1)!}$ up to 3 decimal places. Let $b_n = \frac{(3)^{2n+1}}{(2n+1)!}$; Series is convergent by AST. We can use ASET.

For $N=5$: $b_6 = 0.000\,26 < 0.0005$;

$\therefore S_5 = \sum_{n=1}^{5} \frac{(-1)^n(3)^{2n+1}}{(2n+1)!} = \underline{0.141}$ has error ≤ 0.0005;

(4) $\sum_{n=1}^{\infty} ne^{-n}$ with error $\leq 10^{-3} = 0.001$;

Let $f(x) = xe^{-x}$;
① $f(x)$ is positive on $x \geq 1$;
② $f(x)$ is continuous on $\mathbb{R}$;
③ $f'(x) = -xe^{-x} + e^{-x} = (-x+1)e^{-x}$ is negative on $x \geq 2$;
$\therefore f(x)$ is decreasing on $x \geq 2$;

$\lim_{x\to\infty} \int xe^{-x}dx \overset{*}{=} \lim_{x\to\infty}\left[-xe^{-x} + \int e^{-x}dx\right] = \lim_{x\to\infty}\left[-xe^{-x} - e^{-x}\right] = 0$; $*\left[\begin{array}{ll} u = x; & dv = e^{-x}dx; \\ du = dx; & v = -e^{-x}; \end{array}\right]$

$\therefore 0 < S - U_N < f(N+1)$ with $U_N = \sum_{n=1}^{N} f(n) + \int_{N+1}^{\infty} f(x)\,dx$;

Using a calculator: $f(x+1) = 0$ when $x = 10.118$; Choose $N = 11$;

Then, $U_N = \sum_{n=1}^{11} ne^{-n} + \int_{12}^{\infty} ne^{-n} = \sum_{n=1}^{11} ne^{-n} + (0) - \left[-12e^{-12} - e^{-12}\right] = \boxed{\sum_{n=1}^{11} ne^{-n} + 13e^{-12} = 0.9206}$;

$\boxed{\text{Q3}}$! Find power series representations.

Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on $(-1,1)$; $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ on $\mathbb{R}$; $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ on $\mathbb{R}$;

(1) $f(x) = \arctan(x)$;

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ on } (-1,1) \text{ since } x \in (-1,1) \Rightarrow -x^2 \in (-1,1);$$

$$\arctan(x) = \int \frac{1}{1-(-x^2)}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n} \text{ on } (-1,1);$$

$\arctan(0) = 0 = C$;

$$\boxed{\therefore \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n} \text{ for } x \in (-1,1);}$$

(2) $g(x) = \frac{1}{(1-x)^2}$;

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left[(1-x)^{-1}\right] = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2};$$

$$\text{Then, } g(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \boxed{\sum_{n=1}^{\infty} n x^{n-1}} \text{ on } (-1,1);$$

↑ index shifts by 1.

(3) $h(x) = \frac{x^3}{1+x^2}$; $h(x) = x^3\left(\frac{1}{1-(-x^2)}\right) = (x^3)\sum_{n=0}^{\infty} (-x^2)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n+3}}$ on $(-1,1)$;

(4) $a(x) = x^2\sin(x)$; $a(x) = (x^2)\sin(x) = (x^2)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+3}}$ on $\mathbb{R}$;

(5) $b(x) = \int \cos(x^2)\,dx$; $b(x) = \int \cos(x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\,dx = \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(2n+1)} x^{2n+1}}$ on $\mathbb{R}$;

(6) $c(x) = \frac{d}{dx}\left(\frac{x^2}{1-x^3}\right)$; $\frac{x^2}{1-x^3} = (x^2)\left(\frac{1}{1-(x^3)}\right) = (x^2)\sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n+2}$ on $(-1,1)$;

$$c(x) = \frac{d}{dx}\left(\frac{x^2}{1-x^3}\right) = \boxed{\sum_{n=1}^{\infty} (3n+2)x^{3n+1}} \text{ on } (-1,1);$$

↑ index has to shifted up.

$\boxed{\text{Q4.}}$ Determine the radius of convergence R and the interval of convergence I.

(1) $\sum_{n=0}^{\infty}(3x-2)^n$ is a geometric series and converges if and only if $|r| = |3x-2| < 1$;

Then, $|3x-2| = 3\left|x-\frac{2}{3}\right| < 1$; $\left|x-\frac{2}{3}\right| < \frac{1}{3}$; $\therefore \boxed{R = \frac{1}{3}}$

$\left|x-\frac{2}{3}\right| < \frac{1}{3}$; $-\frac{1}{3} < x-\frac{2}{3} < \frac{1}{3}$; $-\frac{1}{3}+\frac{2}{3} = \frac{1}{3} < x < \frac{1}{3}+\frac{2}{3} = \frac{3}{3} = 1$;

$\boxed{\therefore I = \left(\frac{1}{3}, 1\right);}$ No need to check the endpoints since the geometric series test covers that.

(2) $\sum_{n=1}^{\infty}\frac{x^n}{2n+1}$; Use the Ratio Test; let $a_n = \frac{x^n}{2n+1}$;

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2n+3}\cdot\frac{2n+1}{x^n}\right| = |x|\lim_{n\to\infty}\frac{2n+1}{2n+3} = |x| < 1$; $\therefore \boxed{R = 1;}$

For the interval, check the endpoints: $x = \pm 1$;

For $x = 1$: $\sum_{n=1}^{\infty}\frac{(1)^n}{2n+1} = \sum_{n=1}^{\infty}\frac{1}{2n+1}$; let $a_n = \frac{1}{2n+1}$ and $b_n = \frac{1}{n}$; $L = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$;

Since $0 < L < \infty$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ diverges by the LCT;

For $x = -1$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$; let $b_n = \frac{1}{2n+1}$. Then,

① for $n \geq 1$: $b_n > 0$;

② for $n \geq 1$: $2n+3 > 2n+1 > 0$ and $b_{n+1} = (2n+3)^{-1} < (2n+1)^{-1} = b_n$;

③ $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n+1} = 0$; $\therefore \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$ converges by AST.

$\boxed{\therefore I = [-1, 1);}$

(3) $\sum_{n=1}^{\infty}\frac{x^{2n+1}}{(n+1)!(2n+1)}$; Use the Ratio Test; let $a_n = \frac{x^{2n+1}}{(n+1)!(2n+1)}$;

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+2)!(2n+3)}\cdot\frac{(n+1)!(2n+1)}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|x^2\frac{(2n+1)}{(n+2)(2n+3)}\right| = 0$;

Since $L < 1$ for all $x \in \mathbb{R}$: $\boxed{R = \infty \text{ and } I = \mathbb{R};}$

Q5. Approximate the Integrals.

(1) $A_1 = \int_0^1 x^2 e^{-x^2}\,dx$; Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$, i.e. radius of conv. $R = \infty$;

Soln: $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ and $x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2}$ on $\mathbb{R}$;

Since $[0,1] \subseteq \mathbb{R}$: $\int_0^1 x^2 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n+2}\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{1}{2n+3} x^{2n+3}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)}$; We have an alternating series.

Let $b_n = \frac{1}{n!(2n+3)}$; Then, ① For $n \geq 0$: $b_n > 0$;
② for $n \geq 0$: $(n+1)!(2n+5) > n!(2n+5) > n!(2n+3) > 0$
and $b_{n+1} = \frac{1}{(n+1)!(2n+5)} < \frac{1}{n!(2n+3)} < b_n$;

③ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!(2n+3)} = 0$;

The Alternating Series Estimation Theorem applies: $|S - S_N| < b_{N+1}$;

We want to find N such that $b_{N+1} = \frac{1}{(N+1)!(2N+5)} < \frac{1}{2}(10^{-3}) = 0.0005$;

Since b_n is decreasing, find N by brute force. For $N=4$: $b_5 = 0.00064 \not< 0.0005$;
For $N=5$: $b_6 = 0.00009 < 0.0005$; ✓

$$\boxed{\therefore S_5 = \sum_{n=0}^{5} \frac{(-1)^n}{n!(2n+3)} = 0.189 \approx A_1 \text{ up to 3 decimal places.}}$$

(2) $A_2 = \int_0^1 \sin(x^2)\,dx$; Use $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ on $\mathbb{R}$;

Soln: $\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$ on $\mathbb{R}$;

Since $[0,1] \subseteq \mathbb{R}$: $A_2 = \int_0^1 \sin(x^2)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{x^{4n+3}}{4n+3}\right]_0^1 dx$

$= \sum_{n=0} \frac{(-1)^n}{(2n+1)!(4n+3)}$; ← This is an alternating series.

Let $b_n = \frac{1}{(2n+1)!(4n+3)}$; ① For $n \geq 0$: b_n is positive;
② For $n \geq 0$: Since $\frac{1}{(2n+3)!} < \frac{1}{(2n+1)!}$ and $\frac{1}{4n+7} < \frac{1}{4n+3}$, $b_{n+1} < b_n$;
③ $\lim_{n\to\infty} b_n = 0$;

We can apply the Alternating Series Estimation Theorem: $|S - S_N| < b_{N+1}$;

Find N s.t. $b_{N+1} < \frac{1}{2}(10^{-3}) = 0.0005$; For $N=1$: $b_2 = 0.00076$;
for $N=2$: $b_3 = 0.00001 < 0.0005$; ✓

$$\therefore S_2 = \boxed{\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!(4n+3)} = 0.310} \approx A_2 \text{ up to 3 decimal places.}$$

(3) $A_3 = \int_0^2 \cos(x^2)\,dx$ with error ≤ 0.001;

$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ on $\mathbb{R}$; $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$ on $\mathbb{R}$;

$A_3 = \int_0^2 \cos(x^2)\,dx = \int_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left[\frac{x^{4n+1}}{4n+1}\right]_0^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{4n+1}}{(2n)!(4n+1)}$;

Let $b_n = \frac{(2)^{4n+1}}{(2n)!(4n+1)}$; [Check that $\sum (-1)^n b_n$ converges by AST]

(1) For $n \geq 0$: $b_n > 0$;

(2) $b_{n+1} = \frac{2^{4n+5}}{(2n+2)!(4n+3)} = \frac{2^4}{(2n+2)(2n+1)} b_n$; For $n \geq 2$, $\frac{2^4}{(2n+2)(2n+1)} < 1$ and $b_{n+1} < b_n$;

(3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2^{4n+1}}{(2n)!(4n+1)} = \lim_{n\to\infty}\left(\frac{2^{4n+1}}{(2n)!}\right)\lim_{n\to\infty}\left(\frac{1}{4n+1}\right) = 0$ since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all $x \in \mathbb{R}$;

We can apply the Alternating Series Estimation Theorem for $N \geq 2$: $|S - S_N| < b_{N+1}$;

Find $N \geq 2$ s.t. $b_{N+1} = \frac{1}{(2n+2)!(4n+5)} \leq 0.001$; For $N=5$: $b_6 = 0.0028$
For $N=6$: $b_7 = 0.00021 < 0.001$; ✓

$$\boxed{\therefore S_6 = \sum_{n=0}^{6} \frac{(-1)^n 2^{4n+1}}{(2n)!(4n+1)} = 0.4617 \text{ has error} \leq 0.001;}$$

(4) $A_4 = \int_0^{\frac{3}{4}} \arctan(x^2)\,dx$ with error ≤ 0.0001; Use $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$;

On $x \in (-1,1)$: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; Since $-x^2 \in (-1,1)$ for $x \in (-1,1)$: $\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ on $(-1,1)$;

Then, $\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ on $(-1,1)$, i.e. same radius of conv.

Since $\arctan(0) = 0$: $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$; $\arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}$ also on $(-1,1)$;

Since $[0,\frac{3}{4}] \subseteq (-1,1)$: $A_4 = \int_0^{\frac{3}{4}} \arctan(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^{\frac{3}{4}} x^{4n+2}\,dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(4n+3)}\left(\frac{3}{4}\right)^{4n+3}$;

Let $b_n = \frac{1}{(2n+1)(4n+3)}\left(\frac{3}{4}\right)^{4n+3}$; A_4 converges by AST and we can apply the estimation theorem.
For $N=3$: $b_4 = 0.000025 < 0.0001$; ✓

$$\boxed{\therefore S_3 = \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)(4n+3)}\left(\frac{3}{4}\right)^{4n+3} = 0.13491 \approx A_4 \text{ within } 0.0001;}$$

(5) $A_5 = \int_0^{\frac{2}{3}} x\ln(1+x^3)\,dx$ up to 5 decimal places, i.e. error $\leq \frac{1}{2}(10^{-5}) = 0.000005$;

$A_5 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(3n+5)}\left(\frac{2}{3}\right)^{3n+5}$; $S_3 = \sum_{n=0}^{3} \frac{(-1)^n}{(n+1)(3n+5)}\left(\frac{2}{3}\right)^{3n+5} = 0.02419$ is accurate to 5 decimal places.

(6) $A_6 = \int_0^1 e^{-x^2}\,dx$ with error ≤ 0.005; See lecture notes.